

General limitations on trajectories suitable for super-Penrose process

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Collisions of particles near a rotating black hole can lead to unbound energies $E_{c.m.}$ in their centre of mass frame. There are indications that the Killing energy of debris at infinity can also be unbound for some scenarios of collisions near the extremal black hole horizon (so-called super-Penrose process). They include participation of a particle that (i) has generic (not fine-tuned) parameters and (ii) moves away from a black hole before collision. We show that for any finite particle's mass, such a particle cannot be obtained as a result of the preceding collision. However, this can be done if one of initial infalling particles has the mass of the order N^{-2} that generalizes previous observations made in literature for radial infall in the Kerr background.

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I. INTRODUCTION

Nowadays, high energy particles collisions near black holes attract much attention. In doing so, there are two related but different issues. The first one consists in the question as to under what conditions high (even unbound) energies $E_{c.m.}$ in the centre of mass frame can be obtained. Let the parameters of one of particles be fine-tuned (case (i)). Then, this can give rise to the so-called Bañados-Silk-West (BSW) effect [1]. In case (ii), no fine-tuning is required [2] - [4]. There is also one more case (iii), when fine-tuning of parameters itself

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is the consequence of the fact that a particle moved on a circular orbit, provided this orbit is situated near the horizon of a near-extremal black hole [5], [6].

The second issue concerns the properties of debris after collision. The central question here is whether and under what conditions one can detect at infinity high Killing energy E . The main difficulty here is that a strong redshift can "eat" even a significant gain of the energy obtained in the collision. It turned out that the type of scenario between colliding particles is important here as well. For case (i), when the BSW effect occurs, it was shown that there exist upper limits on E [7] - [9]. Meanwhile, quite recently, a series of works have appeared in which case (ii) was investigated. It implies that the main effect comes from collision between an ingoing and outgoing particles near the horizon (head-on collision, as far as the radial motion is concerned). As was shown in [10] numerically, this type of scenario can lead to a significant increase of E . In doing so, the outgoing particle was obtained as a result of reflection of another ingoing particle from the potential barrier. Then, the outgoing particle under discussion turns out to be close to the fine-tuned one, provided reflection occurred near the horizon. In [11], an outgoing generic particle was taken as an initial condition *per se*. As a result, the authors obtained numerically unbound E . The latter property is called in [11] super-Penrose process. It was pointed out in [12] that this leads to some limitations on the mass m_* of the ingoing particle that should be very large (formally diverging) when the point of collision approaches the horizon (see also reply in the revised preprint version of [11]). Head-on collision with participation of a generic (not fine-tuned) particle of a finite arbitrary mass was considered analytically in [13]. It was shown there that unbound $E_{c.m.}$ are indeed possible in such a process.

As the outcome of collision is very sensitive to the type of scenario and properties of particles before collision, it is important to understand, what kinds of particles are suitable for the super-Penrose process in general. The presence of a so-called usual (not fine-tuned) outgoing particle near the horizon is an essential ingredient for collisions considered in [11] - [13].

In principle, one can impose this initial condition by hand. However, if a particle moves away from the horizon of a black hole, this looks quite unusual. Therefore, it is desirable to find physical justification for such trajectories and learn, what such trajectory can come from. More precisely, we have to elucidate, whether or not they can be obtained from preceding particle collisions. We consider this issue and show that this is impossible. The

result is valid for any number of particles participating in the collision near the horizon of the extremal black hole. It is obtained in a model-independent way, so it does not depend on the particular form of the metric. Additionally, we generalize the observation made in [12] and obtain the value of m_* for a quite generic metric and without additional assumption about radial motion of infalling particles.

It is also worth mentioning that for the extremal Reissner-Nordström black hole, unbound $E_{c.m.}$ can be obtained easily even in the standard BSW picture, without initially outgoing particles [14], [15]. However, for astrophysical purpose, this metric is irrelevant.

Throughout the paper, the fundamental constants $G = c = 1$.

II. BASIC FORMULAS

We consider the axially symmetric metric of the form

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2, \quad (1)$$

where the metric coefficients do not depend on t and ϕ . Correspondingly, the energy $E = -mu_0$ and angular momentum $L = mu_\phi$ are conserved. Here, m is a particle's mass, $u^\mu = \frac{dx^\mu}{d\tau}$ being the four-velocity, τ the proper time along a trajectory. We restrict ourselves to motion within the equatorial plane $\theta = \frac{\pi}{2}$. Then, without the loss of generality, we can redefine the radial coordinate in such a way that $A = N$. We do not restrict ourselves to the Kerr or another concrete form of the metric, so the results are quite general. The equations of motion for a geodesic particle read

$$m\dot{t} = \frac{X}{N^2}, \quad (2)$$

$$m\dot{\phi} = \frac{L}{g_\phi} + \frac{\omega X}{N^2}, \quad (3)$$

$$m\dot{r} = \sigma Z, \quad (4)$$

where

$$X = E - \omega L, \quad (5)$$

$$Z = \sqrt{X^2 - N^2(m^2 + \frac{L^2}{g_\phi})}, \quad (6)$$

and the dot denotes differentiation with respect to τ , the factor $\sigma = +1$ or -1 depending on the direction of motion.

As usual, we assume the forward-in-time, condition $\dot{t} > 0$, whence

$$X \geq 0. \quad (7)$$

The equality can be achieved on the horizon only where $N = 0$. Particles with $X_H > 0$ separated from zero are called usual, particles with $X_H = 0$ are called critical. If $X_H \neq 0$ but is small we call a particle near-critical (see next Section for more detailed explanations). Subscript "H" means that the quantity is calculated on the horizon.

III. PROPERTIES OF PARTICLE'S MOTION NEAR THE HORIZON.

In what follows, we consider the extremal horizon. In combination with formulas from the previous section, this gives rise to some quite definite properties of motion near the horizon. Let us consider first usual particles. In the vicinity of the horizon, there is no turning point since $N \rightarrow 0$, $X > 0$, so $Z^2 > 0$. If the impact parameter $b \equiv \frac{L}{E} < \omega_H^{-1}$, the particle can reach the horizon. If $b > \omega_H^{-1}$, this is impossible since it would contradict condition (7). This means that a usual particle can reach the turning point, where $Z = 0$, and bounce back. We are interested in collisions with small N only, so we take into account the situation when the turning point lies close to the horizon. Then, b must be close to the "critical" value ω_H^{-1} . Near the extremal horizon, the requirement of regularity gives rise to the expansion [16]

$$\omega = \omega_H - B_1 N + O(N^2), \quad (8)$$

where B_1 is some model-dependent coefficient. Correspondingly,

$$X = X_H + B_1 L N + O(N^2). \quad (9)$$

There are two clear definitions of usual ($X_H = 0$) and critical ($X_H \neq 0$) particles. For near-critical particles, we required X_H to be small (see above) but have not yet specify more precisely, how small X_H should be. Now, this can be done on the basis of the expansion (9). It is natural to assume that X_H (the value of X on the horizon) has the same order as the second term taken in the point of collision, where $N = N_c$ (hereafter, subscript "c" indicates the point of collision). Otherwise, we return to previous definitions. Indeed, if $X_H \ll B_1 L N$, the contribution from X_H is negligible, and the particle behaves practically

like the critical one. The opposite case, $X_H \gg B_1 L N$, is typical of a usual particle. Only the intermediate case when

$$X_H = D N_c, \quad (10)$$

deserves special attention. Here, D is some constant. Thus we specify our definition of near-critical particles requiring that (10) is valid. In eq. (10) N_c is not a free variable since it is taken only in the point of collision. This is in contrast with eqs. (8), (9) which are valid for arbitrary N , small enough. Then,

$$X_c = N_c(D + B_1 L) + O(N_c^2). \quad (11)$$

If the particle is exactly critical, $D = 0$. According to our definitions, in the point of collision both critical and near-critical particles are similar in the sense that in both cases

$$X_c = O(N_c). \quad (12)$$

It is seen from (6) that

$$Z(N_c) = O(N_c). \quad (13)$$

As a result, for all cases we can write

$$\lim_{N_c \rightarrow 0} Z = X. \quad (14)$$

In particular, for critical particles both sides of (14) are equal to zero. If a particle is near-critical, sending $N_c \rightarrow 0$ we also send X_H to zero since it is adjusted to N_c according to (10). In doing so, a near-critical particle becomes more and more similar to the critical one and, in the limit, turns into it.

IV. TYPE OF OUTGOING PARTICLES FROM COLLISIONS NEAR HORIZON

In this section we extend approach of [8] (see also Sec. V of [9]) to the case of multiple collision. Let several particles initially move toward the horizon and collide in some point near the horizon. We assume that in the point of collision, the total energy and angular momentum are conserved. As a consequence, in this point we have

$$X_{in} = X_{fin}, \quad (15)$$

where X_{in} is the total contribution of initial particles and X_{fin} is the total contribution of the final outcome. Also, we assume the conservation of the radial momentum. If there are p particles before collision and q particles after it, this entails

$$\sum_{i=1}^p \sigma_i Z_i = \sum_{k=1}^q \sigma_k Z_k. \quad (16)$$

Let us consider the limit in which the point of collision approaches the horizon. We want to elucidate, whether or not a usual outgoing particle (that was absent initially) can arise as a result of such a collision. In doing so, all masses m_i and m_k are considered to be finite.

Statement. If in the initial configuration usual outgoing particles are absent, they cannot appear after collision.

Proof. It follows from (16) that in the limit under discussion,

$$\sum_{i=1}^n \sigma_i (X_i)_H = \sum_{k=1}^m \sigma_k (X_k)_H. \quad (17)$$

By assumption, all initial particles are ingoing. Therefore, in the left hand side of (17), only terms with $\sigma_i = -1$ are present. Then, we have

$$-X_{in} = X_{fin}^{(+)} - X_{fin}^{(-)}. \quad (18)$$

Here, $X_{fin}^{(+)}$ and $X_{fin}^{(-)}$ are, correspondingly, contributions into X_{fin} coming from outgoing and ingoing particles,

$$X_{fin} = X_{fin}^{(+)} + X_{fin}^{(-)}. \quad (19)$$

Now, comparing (15), (18) and (19), we arrive at the result

$$X_{fin}^{(+)} = 0. \quad (20)$$

However, according to (7), each term in this sum is nonnegative. Therefore, the equality (20) is possible only in the case in which each of terms is equal to zero. It means that none of outgoing particles after collision can be usual which completes the proof. If there are several critical or near-critical particles initially, the statement still holds true since such particles do not contribute to X_{in} in the limit under discussion.

If, instead of taking the exact limit, we consider small but nonzero N_c , we should use eq. (13) instead of eq. (14). Then, instead of (20) we obtain that

$$X_{fin}^{(+)} = O(N_c). \quad (21)$$

Thus outgoing particles can be near-critical but, again, they cannot be usual.

The results can be generalized for nonequatorial motion, provided that θ - components of velocities are finite. They will enter Z , being multiplied by N^2 [17] and will not change the conclusion.

V. CASE OF SUPERMASSIVE PARTICLE

If some of particles are so heavy that, formally, $m = O(N^{-1})$ or higher, eq. (14) becomes invalid and our proof fails. This is just the situation discussed in [12] (see their eq. 4) and [11]. The authors of Ref. [12] considered, under which conditions an outgoing particle can be obtained near the horizon as a result of collisions of two particles falling from infinity. It was shown in [12] that for the kerr metric the corresponding mass value of one of particles has the order N^{-2} . Now, we generalize this observation for an arbitrary metric of the form (1). What is even more important, this enables us to elucidate, whether or not the main restriction on the mass of the infalling particle found in [12] depends on the details of the process.

Let two particles, 2 and $*$ (we use the same notations as in [12]), move from infinity (or some finite distance) towards the horizon. After collision, particle 3 falls in a black hole and particle X escapes to infinity. Then, the conservation of the radial momentum reads

$$p_2^r + p_*^r = p_3^r + p_X^r. \quad (22)$$

Here, p^r for each particle is taken from (4), so

$$p_2^r = -\sqrt{X_2^2 - N^2(m_2^2 + \frac{L^2}{g_\phi})}, \quad (23)$$

$$p_*^r = -\sqrt{(m_* - \omega_H L_*)^2 - N^2(m_*^2 + \frac{L_*^2}{g_\phi})}, \quad (24)$$

$$p_X^r = \sqrt{X_X^2 - N^2(m_X^2 + \frac{L_X^2}{g_\phi})}, \quad (25)$$

$$p_3^r = -\sqrt{(X_2 + X_* - X_X)^2 - N^2(\frac{L_3^2}{g_\phi} + m_3^2)}, \quad (26)$$

where we assume for simplicity that $E_* = m_*$ and took into account the conservation of X (15). All particles are taken to be usual. Then, near the horizon, for small N , one can write

for any particle that

$$Z \approx X - \frac{N^2}{2X} \left(m^2 + \frac{L^2}{g_\phi} \right). \quad (27)$$

Then, after some algebra, we obtain in the main approximation that m_* should be large,

$$m_* = \frac{4X_X}{N^2} + O\left(\frac{1}{N}\right). \quad (28)$$

For the extremal Kerr metric, $N^2 \approx \frac{(r-M)^2}{4M^2}$, where M is the black hole mass, r is the Boyer-Lindquist coordinate. Then, eq. (28) corresponds just to eq. (4) of [12], if one takes into account that . In [11], the large value of m_* was related to the simplifying assumption $L_2 = L_* = 0$ taken in [12]. However, it follows from our consideration that for any finite angular momenta the result $m_* = O(N^{-2})$ is qualitatively the same and, in the main approximation, depends on X_X only.

VI. DISCUSSION AND CONCLUSIONS

Thus we showed that the most simple option for arranging the super-Penrose process near black holes is closed. However, there are several other options. Let us enumerate them. Actually, the present result in combination with some previous works gives a complete classification of potentially suitable states.

A. Collision with a supermassive particle

This includes a very massive particle falling in a black hole in the first collision to obtain a usual escaping particle in the near-horizon region. This is the inevitable price for it.

B. Collision near past horizons (white holes)

A particle that moves near the horizon in the outward direction can correspond to white holes rather than to black ones which suggests one more type of high energy collisions. This circumstance was briefly mentioned in [18]. Detailed consideration of such a process was done in [19]. Now, the particle crosses the horizon with $X_H > 0$ by assumption, so we have a usual particle moving in the outward direction. If one bears in mind the potential astrophysical consequences, there is difficulty here since white holes are probably unstable

(see Sec. 15.2 of [20]). Nonetheless, a white hole (past horizon) is an unavoidable part of the whole picture of an eternal black-white hole. Therefore, the complete and coherent consideration of all possible scenarios should include this case as well.

C. Collisions with (near)critical particles

There is a scenario in which one of ingoing near-critical particles bounces back near the horizon thus turning into an outgoing one and collides afterwards with another ingoing particle. Then, relatively high energies were obtained numerically in [10]. It remains to be seen analytically, when this type of scenario indeed leads to the super-Penrose process.

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